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The Ensemble Quantum State of a Single Particle

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Abstract The derivation of the statistical nature of the quantum mechanical wave function is presented within the formalism of quantum mechanics and the second quantization. The statistical wave function may be derived for non relativistic bosons, non relativistic fermions, and relativistic bosons by employing the commuting field operator $\hat{\psi}(x)$. For relativistic electrons a strictly anticommuting $\hat{\psi}(x)$ must be employed to derive the statistical wave function (spinor). The discussion at the end of the paper aims to show the physical plausibility of a statistical wave function.

Keywords Ensemble · Quantum state · Second quantization · Relativistic ensemble states · Eigenstates · Annihilation operator · Grassmann field

Introduction

The work of Ballantine [1], Einstein [2], Khrennikov [3], Bohr [4], and Rosenstein [5] strongly bring to the attention that, besides being a nontrivial problem, the interpretation of the concept of state in quantum mechanics is imperative to the understanding of quantum phenomena and quantum theory. These in depth investigations have resulted in a multiple of interpretations of the concept of state in quantum mechanics with the most common among the physicists being the statistical interpretation reflecting Einstein's stand on this subject, and the Copenhagen interpretation invented by Bohr. A famous thought experiment which to a good extent reveals the nature of a quantum state is the widely known EPR argument which suggests that the wave function represents an incomplete set of information about a quantum system. However, while the EPR argument defines completeness as a correspondence between elements of the physical reality and elements of the physical theory it somewhat leaves the definition of physical reality open to the reader. This fact is implied in the EPR paper because the reasonable definition of physical reality in the EPR argument is taken to be an element which is preceded by a measurement (with certainty) of a physical quantity and

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thereby making the physical reality comes second to the measurement of the physical quantity. This method of defining physical reality might have provided a window for a counter argument of the EPR argument. However, any legitimate statement about nature should be a consequence of honest investigations and should not be constructed for the purpose of a counter argument only. If the window in the EPR paradox mentioned above indeed formed a weakness in it's presentation then it is possible that the counter argument to the EPR proposal originated from this weakness. If this possibility regarding the motivation for developing a counter argument is confirmed then one can be certain that the counter argument was not a result of scientific observation of nature. It is not the intention here to critically criticize any counter arguments of the EPR "paradox", however, it is worth mentioning that a remarkable comment made [1] on the subjective interpretation of the quantum probability is that it is not a necessary one. In addition, the statistical interpretation have the advantage of not running into what have been called the "metaphysical complications that one would have to confront if one is to believe that a pure state is an exhaustive description of an individual system" [1]. The EPR argument didn't propose a statistical interpretation of the quantum state, nevertheless it lead to such an interpretation [6]. Moreover, quantum interference have been concluded [3] by considering the statistical interpretation of the quantum state.

With many arguments in favor of both the statistical and the subjective interpretation of the quantum state it becomes relevant to present an analytic derivation which will support one of the two interpretations. In what follows it is going to be the statistical interpretation which will be given a proof. For that matter one firstly should discriminate between statistical averages which are evaluated for a system in a mixed state and quantum expectation values which are calculated for a system in a pure state. The statistically interpreted wave function $\psi(x)$ will be proven to be the expectation value of the annihilation operator $\hat{\psi}(x)$ evaluated in a pure state. The pure state is constructed to be an eigenvector of the operator $\hat{\psi}(x)$, where $x = (\vec{x}, t)$. This eigenvector will be designated by the symbol $|\rangle$ and thus one may write $\psi(x) = \langle|\hat{\psi}(x)|\rangle$. The eigenvector of $\hat{\psi}(x)$ will bear a strong resemblance to the eigenvector of the positive frequency electric field operator defined in quantum optics but the interpretation will be entirely different. It was shown [7] that identifying the function $\psi(x)$ defined above with the quantum mechanical wave function exclusively rely on whether or not the operator $\hat{\psi}(x)$ satisfy a dynamical equation similar to the one satisfied by the wave function. Thus if $\hat{\psi}(x)$ satisfy a certain dynamics then the function $\psi(x)$ must satisfy the same dynamics.[1] For non relativistic spinless particles the function $\psi(x)$ will represent the state function if it obeys the Schrödinger equation which means that in that regime the operator $\hat{\psi}(x)$ must satisfy the Schrödinger equation as well. The possibility of utilizing commuting fields to define both bosonic and fermionic ensemble wave functions when in the non relativistic regime will be discussed below in the section on limitations. For relativistic bosons the dynamical equation which the annihilation field operator $\hat{\psi}(x)$ satisfy is known to be the Klein-Gordon equation [8]. In fact only the commuting $\hat{\psi}(x)$ operator satisfy the Klein-Grodon equation. This means that the ensemble wave function $\psi(x)$ pertaining to a relativistic boson is equal to $\langle|\hat{\psi}(x)|\rangle$ with $\hat{\psi}(x)$ being a *strictly* commuting operator. For a charged spinless relativistic boson the ensemble wave function $\psi(x)$ was shown [7] to be proportional to the average occupation number of particles and antiparticles.

The relativistic electron with spin-$\frac{1}{2}$ is a special case. For the relativistic Dirac electron one must identify the function $\psi(x)$ with the spinor that satisfies the Dirac equation. According to the prescription $\psi(x) = \langle|\hat{\psi}(x)|\rangle$ the field operator $\hat{\psi}(x)$ must satisfy the Dirac

[1] This fact is true given the eigenvector $|\rangle$ is independent of space and time.

equation as well. However, the Dirac equation is satisfied only by anticommuting $\hat{\psi}(x)$. Therefore the ensemble spinor $\psi(x)$ pertaining to a relativistic electron is $\langle|\hat{\psi}(x)|\rangle$ with $\hat{\psi}(x)$ a *strictly* anticommuting operator. The commutation rule of the field $\hat{\psi}(x)$ will determine the spin statistics which will in turns determine the structure of the eigenvector $|\rangle$. The eigenvector of an anticommuting operator $\hat{\psi}(x)$ is possible to define only by introducing grassmann numbers. This problem will be treated in Sect. 5.

No less important than a mathematical proof of an ensemble wave function is the physical plausibility of such a description. For that purpose some remarks are made at the end of the paper to propose the physical origin behind the fluctuations that are to occur even for an ensemble of similar quantum particles prepared under the same initial conditions. A brief section is also dedicated to show the optical application of the electron eigenvector equation.

1 The Ensemble Quantum State of a Spinless Non Relativistic Particle

The statistical interpretation of the quantum state, and thus the quantum probability, is possible to infer from an expression which relates the quantum state to a statistical, though not canonical, average. This expression will be derived by means of the formalism of quantum theory and the second quantization (field theoretic).

When defining mean values in quantum theory physicists discriminate between two types of averages. These two types of averages are the statistical average [9] of an observable $\hat{q}$ in a mixed state, and the quantum mechanical expectation value of an observable $\hat{q}$ in a pure state. The statistical average of $\hat{q}$ has been defined [10] to be the incoherent superposition given by

$$\langle\hat{q}\rangle = \sum_i \rho_i \langle\hat{q}\rangle_i \tag{1}$$

where ρ_i is a real number [9, 10] reflecting the statistical weight of the ith pure state. The quantities $\langle\hat{q}\rangle_i$ which appear in (1) represent the coherent quantum expectation value of the observable $\hat{q}$ evaluated in the pure state

$$|\Phi^i\rangle = \sum_k c_k^i |\phi_k\rangle \tag{2}$$

where $\{|\phi_k\rangle\}$ is a set of eigenvectors of a complete operator and where the c_k^i's are complex numbers. It is (2) which should be of utmost concern when one attempts to provide an analytic proof of the statistical nature of the quantum state. For a statistical interpretation of the mixed state which leads to (1) is self evident and do not need any proof. In fact, (1) leads to the result

$$\langle\hat{q}\rangle = \mathrm{Tr}(\hat{\rho}\hat{q}) \tag{3}$$

where $\hat{\rho} = \sum_i \rho_i c_{k'}^{(i)*} c_k^{(i)}$ is the density operator for a system prepared in a mixed state, and Tr stands for trace. Evidently (3) have a strong resemblance to the classical statistical ensemble average [9]. In the literature on quantum mechanics the term state or wave function is used frequently to mean the pure quantum state [10] therefore the word state in this paper shall bare the same meaning as a pure state. The statistical interpretation of the pure state in (2) will be accomplished by means of the complex coefficients c_k^i.

Consider the average of an observable $\hat{q}$ evaluated in a pure state given by (2). This average can be calculated using (1) along with the constraint $\rho_i = 1$, $\rho_j = 0\ \forall j \neq i$, which will reduce the expression in (1) to

$$\langle \hat{q} \rangle = \langle \hat{q} \rangle_i = \langle \Phi^i | \hat{q} | \Phi^i \rangle \tag{4}$$

Denote the expectation value of the observable $\hat{q}$ by q_i such that

$$q_i = \langle \hat{q} \rangle_i = \langle \Phi^i | \hat{q} | \Phi^i \rangle \tag{5}$$

with the normalization condition

$$\langle \Phi^i | \Phi^i \rangle = 1 \tag{6}$$

Equation (5) may be expressed as an eigenvalue equation (multiply from the left by the bra $\langle \Phi^i |$)

$$\hat{q} | \Phi^i \rangle = q_i | \Phi^i \rangle \tag{7}$$

where $\hat{q}|\phi_k\rangle = q_i|\phi_k\rangle$. Now consider the expectation value, as defined by (5), of the field operator $\hat{\psi}(\vec{x})$ which is defined in the second quantization to be the operator that annihilates a particle at the coordinate $\vec{x}$, and denote the pure state operator by $\rho = |\rangle\langle|$. Bringing (7) into the picture one may write the eigenvalue equation

$$\hat{\psi}(\vec{x})|\rangle = \psi(\vec{x})|\rangle \tag{8}$$

where $\psi(\vec{x})$ replaces q_i in (7). Consistency with (5) will provide a method for calculating $\psi(\vec{x})$

$$\psi(\vec{x}) = \langle|\hat{\psi}(\vec{x})|\rangle \tag{9}$$

Now one can claim that the quantum mechanical wave function $\Psi(\vec{x})$ is the normalized form of the function in (9)

$$\Psi(\vec{x}) = \frac{1}{\sqrt{N}}\langle|\hat{\psi}(\vec{x})|\rangle \tag{10}$$

where

$$N = \sum_k \langle n_k \rangle \tag{11}$$

is the normalization constant.[2] In quantum field theory the operator $\hat{\psi}(\vec{x})$ is often expanded in a series of annihilation operators such that

$$\hat{\psi}(\vec{x}) = \sum_k \hat{c}_k \phi_k(\vec{x}) \tag{12}$$

where $\hat{c}_k|n_k\rangle = \sqrt{n_k}|n_k - 1\rangle$, n_k is the number of particles (quanta) per state k, and the ϕ_k's are eigenvectors of the energy operator. A legitimate normalized coherent pure state which satisfies (8) was shown [7] to be a product of the eigenvectors of the annihilation operators

$$|\rangle = \prod_{k=1}^{\infty} |\alpha_k\rangle \tag{13}$$

[2]The validity of the constraint in (11) for a system in a pure state will be discussed in Sect. 2, and its validity for statefunctions that are grassmann fields will be discussed briefly in the section on the ensemble quantum state for a relativistic electron.

where

$$\hat{c}_k|\alpha_k\rangle = \alpha_k|\alpha_k\rangle \tag{14}$$

and where the eigenvectors are given by [11]

$$|\alpha_k\rangle = \sum_{n_k=0}^{\infty} \frac{\alpha_k^{n_k}}{\sqrt{n_k!}} \mathrm{e}^{-|\alpha_k|^2/2}|n_k\rangle \tag{15}$$

Note that the square of the coefficients of the expansion in the Fock basis given in (15) is precisely the Poisson distribution.[3] The eigenvalues of (14) are given by[4] $\alpha_k = \sqrt{\langle n_k\rangle}\,\mathrm{e}^{i\delta_k}$, where δ_k is a phase factor corresponding to the complex value of α_k. Now with the expansion in (12) one will immediately get the result that the function in (9) can be written as an expansion with the expansion coefficients being the eigenvalues of the operators $\hat{a}_k$

$$\psi(\vec{x}) = \sum_k \alpha_k\phi_k(\vec{x}) \tag{16}$$

which means that the wave function in (10) can be written as

$$\Psi(\vec{x}) = \sum_k \left(\frac{\langle n_k\rangle}{N}\right)^{1/2} \mathrm{e}^{i\delta_k}\phi_k(\vec{x}) \tag{17}$$

When using the ϕ_k's which are the solutions to the time dependent Schrödinger equation one can write the normalized function $\Psi(\vec{x})$ in it's time dependent form

$$\Psi(x) = \sum_k c_k\phi_k(x) \tag{18}$$

where the vector notation $x = (\vec{x}, t)$ is used and where $c_k = (\langle n_k\rangle/N)^{1/2}\mathrm{e}^{i\delta_k}$. If one identifies $\Psi(x)$ as the pure state then a comparison between (18) and (2), with the understanding that (2) must be written in the coordinate representation, will make clear the ensemble interpretation of the state function for a single particle. Identifying $\Psi(x)$ with the state function may be done by proving that $\Psi(x)$ satisfies the Schrödinger equation. Such a proof is facilitated by the fact that the field operator $\hat{\psi}(x,t)$ (for both bosons and fermions) itself satisfies the Schrödinger equation and the fact that the state $|\rangle$ is independent of position and time.[5] The ensemble interpretation of the quantum state $\Psi(x)$ can be infered from (18) where the single particle state is directly proportional to the average particle number (average number of quanta) per state $\langle n_k\rangle$. According to the ensemble interpretation [1] the $\langle n_k\rangle$'s appear in the sum over the quantum numbers because in an ensemble of similarly prepared systems, say the set of all electrons, which have been subjected to the same state preparation one may not control the number of the members in the ensemble which contribute to the kth quantum state. Thus the state function $\Psi(x)$ for a single particle may be identified as a purely

[3] A simple derivation of the eigenvector in (15) is given in reference [12].

[4] It is very easy [7] to show that $\langle n_k\rangle = |\alpha_k|^2$, where in evaluating the average the Poisson distribution pertaining to the coefficients of the expansion in (15) is used.

[5] The proof that the function $\Psi(x)$ satisfies the Schrödinger equation is straightforward, see for example reference [7].

statistical function which may be utilized to calculate quantum expectation values. This interpretation of $\Psi(x)$ is in full support of Einstein's approval for an ensemble interpretation of the quantum state from which one deduces the incompleteness of a quantum state vector in describing an individual system.

In agreement with Ballantine's assertion regarding the ensembles being contemplated in the statistical interpretation of the quantum state versus the ensembles used in statistical mechanics, one notes that the ensemble represented by the wave function $\Psi(x)$ in (18) is different from the canonical ensembles defined in statistical mechanics. The difference between the two ensembles may be seen by the fact that when evaluating the average $\langle n_k\rangle$ one must use the Poisson distribution [7] and not a distribution which correspond to any of the canonical distributions in statistical mechanics. A quantum ensemble analogous to the classical thermodynamic ensemble is possible only when evaluating the average, such as the average shown in (1), of an observable of a system which is in a mixed state.

Since the constraint in (11) is similar to the one in the grand canonical ensemble theory [9] and since, as discussed above, the ensemble considered here is unlike any of the canonical ensembles it is important to present a direct proof of the constraint in (11).

2 Proof of the Normalization Condition

One must not confuse the $\langle n_j\rangle$ which appears in (11) with the average occupation number defined in the grand canonical ensemble theory. While it is true that $N=\sum_j\langle n_j\rangle_{\mathrm{gc}}$ in the grand canonical ensemble theory the fact is when evaluating $\langle n_j\rangle_{\mathrm{gc}}$, say for Bosons,[6] one needs to use the geometric distribution function [9]

$$P_B(n_j)=\frac{(\langle n_j\rangle)^{n_j}}{(\langle n_j\rangle+1)^{n_j+1}} \tag{19}$$

however, in (11) the quantity $\langle n_j\rangle$ is evaluated using the Poisson distribution function [7]

$$P(n_j)=\frac{(\langle n_j\rangle)^{n_j}\,\mathrm{e}^{-\langle n_j\rangle}}{n_j!} \tag{20}$$

and thus the validity of the constraint $N=\sum_j\langle n_j\rangle_{\mathrm{gc}}$ when $\langle n_j\rangle$ is in place of $\langle n_j\rangle_{\mathrm{gc}}$ will need a justification.

Consider the observable

$$\hat{n}(x)=\hat{\psi}^\dagger(x)\hat{\psi}(x) \tag{21}$$

which is defined in the second quantization to be the particle number density operator. Next take the average of $\hat{n}(x)$ as in (1) and take the density operator for a pure state to be $\rho=|\rangle\langle|$

$$\langle\hat{n}(x)\rangle=\mathrm{Tr}(\rho\hat{n}(x)) \tag{22}$$

Equation (22) may be written as

$$\langle\hat{n}(x)\rangle=\mathrm{Tr}\big(\rho\hat{\psi}^\dagger(x)\hat{\psi}(x)\big)=\mathrm{Tr}\big(\hat{\psi}(x)\rho\hat{\psi}^\dagger(x)\big) \tag{23}$$

[6]For fermions the distribution is given by $P_F(n_j)=(\langle n_j\rangle)^{n_j}/(1-\langle n_j\rangle)^{n_j-1}$ with n_j being equal to either 1 or 0.

which is true due to the cyclic property of the trace. Now evaluate the trace in (23) using the coherent pure state and the ρ given above. This will immediately give

$$\langle \hat{n}(x)\rangle = \psi^*(x)\psi(x) \tag{24}$$

where in getting (24) use have been made of the adjoint of (8). Now substitute the function in (16) into the right hand side of (24)

$$\langle \hat{n}(x)\rangle = \sum_{j,k}\langle n_j\rangle^{1/2}\langle n_k\rangle^{1/2}\,\mathrm{e}^{i(\delta_k-\delta_j)}\phi_j^*(x)\phi_k(x) \tag{25}$$

For $j=k$ (25) will reduce to the common form for the density matrix in a multiparticle system [8]

$$\langle \hat{n}(x)\rangle = \sum_{j}\langle n_j\rangle\phi_j^*(x)\phi_j(x) \tag{26}$$

In the second quantization the total particle number operator is obtained by integrating the number density operator $\hat{n}(x)$ over all space

$$\hat{N} = \int d^3x\hat{n}(x) \tag{27}$$

Equation (27) will lead to its ensemble average form

$$N = \langle \hat{N}\rangle = \int d^3x\ \langle \hat{n}(x)\rangle \tag{28}$$

With the orthonormal property of the set $\{\phi_j(x)\}$ it becomes evident from (28) and (25) that

$$N = \sum_j \langle n_j\rangle \tag{29}$$

Therefore imposing the particle number constraint in (11), which was assumed before [7] to be true in order to demonstrate the compatibility of the norm square of the function in (17) with the probability density function, is justified.

3 Optical Application

Realizing the similarity in formalism between the non relativistic particle field eigenvector equation, (8), and the electric field operator eigenvector equation [12] it becomes natural to question if by means of (8) one may provide a theoretical description for the appearance of fringes when conducting a double slit experiment with electrons. The appearance of fringes being a phenomenon that have been observed for both particle beams and light beams and one that was explained by the concept of coherence. Coherence may quantitatively be described by the normalized correlation function [13]

$$g^{(1)}(x,x') = \frac{G^{(1)}(x,x')}{\sqrt{G^{(1)}(x,x)G^{(1)}(x',x')}} \tag{30}$$

where x again indicates both the spatial and the temporal coordinates and where $G^{(1)}(x,x')$ is the correlation function which in quantum optics is defined to be

$$G^{(1)}(x,x') = \mathrm{Tr}\big(\rho \hat{E}^{(-)}(x)\hat{E}^{(+)}(x')\big) \tag{31}$$

with $\hat{E}^{(+)}(x) = \hat{\mathrm{e}}^* \cdot \hat{\vec{E}}^{(+)}(x)$, $\hat{\mathrm{e}}$ a complex unit vector representing the direction of photon polarization, and $\hat{\vec{E}}^{(+)}(x)$ the photon annihilation operator [14]. Likewise, $\hat{E}^{(-)}(x) = \hat{\mathrm{e}} \cdot \hat{\vec{E}}^{(-)}(x)$ where $\hat{\vec{E}}^{(-)}(x)$ is the photon creation operator. The superscript (1) in the function $G^{(1)}(x,x')$ indicates that it is the first-order correlation function. Furthermore, the normalized correlation function is related to the visibility of fringes with a value of unity indicating the appearance of fringes [15].

The photon annihilation and creation operators are based on separating the electric field operator into a positive and negative frequency parts such that

$$\hat{\vec{E}}(x) = \hat{\vec{E}}^{(+)}(x) + \hat{\vec{E}}^{(-)}(x) \tag{32}$$

and

$$\hat{\vec{E}}^{(-)}(x) = \hat{\vec{E}}^{(+)\dagger}(x) \tag{33}$$

making it possible to write the correlation function in (31) in terms of the annihilation operator and it's adjoint

$$G^{(1)}(x,x') = \mathrm{Tr}\big(\rho \hat{E}^{(+)\dagger}(x)\hat{E}^{(+)}(x')\big) \tag{34}$$

Now a necessary condition for a first-order coherence, and thus the appearance of fringes, is

$$|g^{(1)}(x,x')| = 1 \tag{35}$$

which is an equality that may be fulfilled by the factorization condition [13]

$$G^{(1)}(x,x') = E^*(x)E(x') \tag{36}$$

where the quantities on the right hand side (36) are the electric field and its complex conjugate. In quantum optics the electric field and it's complex conjugate are related to the photon annihilation and creation operators, respectively. The relation between the classical and the quantum fields is expressed by means of the eigenvector equations

$$\hat{E}^{(+)}(x)|\rangle = E(x)|\rangle \tag{37}$$

and

$$\langle|\hat{E}^{(+)\dagger}(x) = \langle|E^*(x) \tag{38}$$

Clearly, if the photon is in a state such that $\rho = |\rangle\langle|$ then the correlation function in (34) will factorize into a form similar to the one in (36). Thus if the photon is in the ρ state then it will satisfy the first-order coherence condition in (35) and therefore the appearance of fringes.

The above argument regarding the appearance of fringes was possible to make due to the eigenvector equation in (37) and it's adjoint. Equation (8) is the particle eigenvector equation which is analogous to the electric field eigenvector equation with the field operator

$\hat{\psi}(x)$ in place of the electric field operator $\hat{E}^{(+)}(x)$ and the eigenfunction $\psi(x)$ in place of the function $E(x)$. One may carry the analogy further as (21) and (22) prove that the ensemble average of the observable $\hat{n}(x)$ is in fact the correlation function defined in (34) but with $\hat{\psi}(x)$ and $\hat{\psi}^{\dagger}(x)$ in place of $\hat{E}^{(+)}(x)$ and $\hat{E}^{(+)\dagger}(x)$. Therefore, for electrons the correlation function which corresponds to (34) is

$$G^{(1)}(x,x') = \mathrm{Tr}\big(\rho\hat{\psi}^{\dagger}(x)\hat{\psi}(x')\big) \tag{39}$$

which with (8) and it's adjoint

$$\langle|\hat{\psi}^{\dagger}(x) = \langle|\psi^{*}(x) \tag{40}$$

may factorize into a form similar to (36)

$$G^{(1)}(x,x') = \psi^{*}(x)\psi(x') \tag{41}$$

thus making the normalized correlation function equal to unity as in (35) and that will indicate the appearance of fringes for the electrons that are in the state described by $\rho = |\rangle\langle|$. Therefore the appearance of electron fringes is possible to describe theoretically by means of defining an eigenvector equation for the electron annihilation field operator as in (8).

4 Limitations

A strict condition on the function $E(x)$ in (37) is the fact that it must satisfy the wave equation [13]. Since the eigenvector in (37) is independent of space and time this restriction will further demand the operator $\hat{E}^{(+)}(x)$ itself to satisfy the wave equation. This fact is evident by multiplying (37) from the left by the bra vector $\langle|$ and utilizing the normalization $\langle|\rangle = 1$. The particular constraint on the electric field operator mentioned above clearly applies to the electron field operator in (8). In the case of the electron the constraint becomes the fact that the function $\psi(x)$ in (8) must obey the Schrödinger equation and that will immediately demand the operator $\hat{\psi}(x)$ in (8) to obey the Schrödinger equation as well. On the other hand the operator $\hat{\psi}(x)$, being time dependent, must satisfy the Heisenberg equation. The Heisenberg equation relates the evolution of the operator to it's commutator with the Hamiltonian operator which for both the commuting and the anticommuting field operators lead to the Schrödinger equation for $\hat{\psi}(x)$. This fact along with (9), and the fact that $|\rangle$ is independent of time and coordinate, will make the function $\psi(x)$ be the formal solution to the Schrödinger equation. Without the constraint of $\hat{\psi}(x)$ satisfying the Schrödinger equation the function in (17) may not be identified with the quantum mechanical state function. The question of whether to use commuting or anticommuting field operators did not matter in as far as the derivation of the non relativistic spinless ensemble quantum state function given in (18). The critical equation which will lead to (18) is (14), *i.e.*, the eigenvector equation for the annihilation operator. For bosons the eigenvectors are given by (15) which clearly admits infinitely many particles (quanta) into a single k state and thus rules out the possibility of defining (8), which leads to (14) and (18), for fermions. In deriving the ensemble state function in (18) for non relativistic electrons the problem may be classified as being two fold:

1. The possibility of defining an eigenvector similar to the one in (15) for a particle which obeys the fermion statistics.

2. The presentation of a proof that the statistical function $\Psi(x)$ defined for electrons is a solution to the Schrödinger equation.

The discussion above clears item number 2 out of the way but for item number 1 a reinterpretation of the state in (15) becomes necessary. It was shown in a former paper [7] that defining an eigenvector as in (15) for electrons demanded a model in which there exist an infinite number of orbitals per a given quantum state k. Each orbital was assumed to be occupied by either one electron or non. This model may be represented by the sequence

$$A_k = (n_k^1, n_k^2, n_k^3, \dots, n_k^i, \dots) \tag{42}$$

where n_k^i is the number of electrons in the kth state found in the ith slot. Clearly n_k^i can hold only two values which are either 1 or 0. The position of the slot may represent the experimental trial number. Thus n_k^i may be read as the number of electrons found in the kth state at the ith run of the experiment (either 1 or 0). Thus the total number of elements in the sequence A_k may be interpreted as the number of experimental runs to find if an electron does occupy the kth state or not. This sequence of experiments, measuring if an electron occupies a kth state or not, can be identified with Bernoulli trials. It is known [16] that in the limit of an infinitely many trials the Bernoulli distribution approaches the Poisson distribution. Thus if p_k is the probability of finding an electron in the kth state at any of the experimental runs and if q_k is the probability of not finding one in the kth state in the same run then

$$P(n_k) = \frac{j_k!}{n_k!(j_k - n_k)!} p_k^{n_k} q_k^{(j_k - n_k)} \tag{43}$$

is the probability that out of the[7] $j_k = |A_k|$ experiments n_k of them[8] will find an electron in the kth state. The limit[9] $j_k \gg 1$ may be interpreted as conducting the experiments infinitely many times. In that limit the probability in (43) may be approximated by a Poisson distribution similar to the one given in (20). One may now contemplate an eigenvector for the annihilation operator which describes the above experimental trials. The general eigenvector for the kth state may be constructed as follows

$$|\beta_k\rangle = \sum_{n_k=0}^{\infty} C_{n_k} |n_k\rangle \tag{44}$$

with

$$C_{n_k} = \frac{(\langle n_k \rangle \mathrm{e}^{i\delta_k})^{n_k/2}}{\sqrt{n_k!}} \mathrm{e}^{-\langle n_k \rangle/2} \tag{45}$$

and $|C_{n_k}|^2$ as the probability that out of a total of j_k runs a subtotal of n_k times an electron is found in the kth state. The eigenvector in (44) leads to the ensemble electron state function given in (18), the statistical interpretation of which was discussed following (18). The model of interpreting the n_k as the number of experimental runs that find an electron in the kth state

[7] For any sequence A the symbol $|A|$ indicates the cardinality of the A, or the total number of elements in A.

[8] Given the set $B = \bigcup_{\substack{i \\ n_k^i \neq 0}} n_k^i$ then one may define $n_k = |B|$.

[9] This limit should be supplemented by the condition that the product $j_k p_k$ is moderate. It is also the same limit imposed by Khrennikov when he discusses the frequency interpretation of the quantum probability.

necessitates the usage of a commuting field operator $\hat{\psi}(x)$. The field operator $\hat{\psi}(x)$ do not create an electron, in fact it creates an electron filled quantum state k in any of the slots of the sequence A_k. The general eigenvector of $\hat{\psi}(x)$ is constructed by forming a product between the states that were defined in (44)

$$|\rangle = \prod_{k=1}^{\infty} |\beta_k\rangle \tag{46}$$

Each of the vectors $|\beta_k\rangle$ expresses the outcome of an experiment on an ensemble of electrons. In the experiment a measurement is made on each electron in the ensemble to determine whether or not it is in the kth state. Clearly the operator $\hat{\psi}(x)$ defined for an ensemble of electrons here is one that is constructed to place as many electrons in the ensemble into a single quantum state k. For the electrons in the ensemble are independent and as such one do not violate the exclusion principle by assuming that many electrons in the ensemble may occupy the same quantum state.

It is possible to object to the statistical interpretation given to the eigenvector in (46) because with a predetermined interpretation of (46) the statistical interpretation of the state function in (18) does not become defined as analytically driven. To give an answer to that possible objection one must recall that (17) was analytically derived for a boson using the formalism of quantum mechanics and the interpretations of the second quantization. With an analytic derivation of the statistical state function pertaining to a boson it becomes logical to expect that the same interpretation is attributable to a fermion state function. For the concept of a state function must be a general one and it may not be interpreted statistically for bosons and reinterpreted for fermions. Thus given that the ensemble wavefunction in (18) was derived for bosons it becomes natural to think about the extension of the ensemble interpretation such that it can include the fermion wavefunction which will justify the anticipated statevector in (46). Having defined a sequence of experiments to detect an electron in the kth state, as in (42), made the employment of the commuting field $\hat{\psi}(x)$ in deriving the ensemble electron wavefunction a possibility and thereby cleared item number 1 mentioned above. If one is to still ask the question; can the function $\psi(x)$, derived by means of the commuting $\hat{\psi}(x)$, indeed be identified with the electron wavefunction? The answer will be yes only if $\psi(x)$ corresponds to a spinless non relativistic electron. For in that case both the commuting and the anticommuting $\hat{\psi}(x)$ will satisfy the Schrödinger equation and that property of $\hat{\psi}(x)$ will prove that the function $\psi(x)$ is the electron wavefunction when the Schrödinger operator is applied to $\hat{\psi}(x)$ in (9).

For the relativistic electron one must strictly use anticommuting $\hat{\psi}(x)$ in (9) to derive the ensemble quantum state for the electron. This limitation is due to the fact that only the anticommuting $\hat{\psi}(x)$ will satisfy the relativistic Dirac electron equation. For that purpose a review of (8) for anticommuting fields becomes necessary.

5 The Ensemble Quantum State for Relativistic Electrons

Extending (17) to include relativistic electrons will demand the study of eigenstates of anticommuting field operators, and that is because the Dirac equation is satisfied by the anticommuting annihilation field operator. However, fermionic eigenstate of the anticommuting annihilation field operator was possible to find by means of grassmann numbers only [17, 18]. The usage of such an eigenstate will finally lead to a wavefunction which is a grassmann field and not a complex field. This route of defining a state function as a grassmann field is mandatory only in the relativistic regime and only if one seeks to connect the relativistic bispinor to the relativistic field operator through an eigenvalue equation of the form given in (8). The derivation of the analogous equation to (17) for relativistic electrons will be presented next.

Up until (65) the analysis will be a semi replica of what was published [17] on the fermionic coherent states the presentation of which is important for clarity only. The relations among fermionic creation and annihilation operators are

$$\{\hat{b}_i, \hat{b}_j^\dagger\} = \delta_{ij} \tag{47}$$

$$\{\hat{b}_i, \hat{b}_j\} = \{\hat{b}_i^\dagger, \hat{b}_j^\dagger\} = 0 \tag{48}$$

where the curly brackets indicate anticommutation and where

$$\hat{b}_j^\dagger|1\rangle = \hat{b}_j|0\rangle = 0 \tag{49}$$

Among the grassmann numbers the anticommutation relations must be defined as well

$$\beta_i^2 = \beta_i^{*2} = \{\beta_i, \beta_j\} = \{\beta_i^*, \beta_j\} = \{\beta_i^*, \beta_j^*\} = 0 \tag{50}$$

The relations among the grassmann numbers with the Fermion creation-annihilation operators are

$$\{\beta_i, \hat{b}_j\} = \{\beta_i, \hat{b}_j^\dagger\} = \{\beta_i^*, \hat{b}_j\} = \{\beta_i^*, \hat{b}_j^\dagger\} = 0 \tag{51}$$

where the grassmann numbers β_i and β_i^* are independent and conjugate to one another. The Fermion coherent state for the kth mode is

$$|\beta_k\rangle = \mathrm{e}^{\hat{b}_k^\dagger\beta_k - \beta_k^*\hat{b}_k}|0\rangle \tag{52}$$

This bears a strong resemblance to the Bosonic coherent state except for the sign of that part of the argument with the creation operator. When expanding the exponential in (52) only terms up to the second power will survive. Because of the set of (50) and (51) terms with power three or above will be equal to zero. Hence the expansion in (52) will be as simple as

$$|\beta_k\rangle = \left(1 + \left(\hat{b}_k^\dagger\beta_k - \beta_k^*\hat{b}_k\right) + \hat{b}_k^\dagger\hat{b}_k\beta_k^*\beta_k - \frac{\beta_k^*\beta_k}{2}\right)|0\rangle = \left(1 - \frac{\beta_k^*\beta_k}{2}\right)|0\rangle - \beta_k|1\rangle \tag{53}$$

which one can prove to be true with the aid of (47), (49), and (51). With the displacement operator in its expanded form it is possible to prove that the state $|\beta_k\rangle$ is an eigenstate of the annihilation operator

$$\hat{b}_k|\beta_k\rangle = -\hat{b}_k\beta_k|1\rangle = +\beta_k\hat{b}_k|1\rangle = \beta_k|0\rangle \tag{54}$$

however, from (53) one may write the vacuum state as a multiple of the state $|\beta_k\rangle$

$$|\beta_k\rangle = \left(1 + \frac{\beta_k\beta_k^*}{2}\right)|0\rangle - \beta_k|1\rangle = |0\rangle + \frac{\beta_k\beta_k^*}{2}|0\rangle - \beta_k|1\rangle \tag{55}$$

Upon multiplying (55) by β_k and using the property of the grassmann numbers in (50) one can verify the relation

$$\beta_k|0\rangle = \beta_k|\beta_k\rangle \tag{56}$$

Equation (56) together with (54) will show that the state $|\beta_k\rangle$ is an eigenstate of the operator $\hat{b}_k$

$$\hat{b}_k|\beta_k\rangle = \beta_k|\beta_k\rangle \tag{57}$$

Likewise the displacement operator

$$\langle 0|\mathrm{e}^{\beta_k^*\hat{b}_k - \hat{b}_k^\dagger\beta_k} = \langle\beta_k| \tag{58}$$

will lead to an eigenvalue equation which is the adjoint of the one that appears in (57)

$$\langle\beta_k| = \langle 0|\left(1 - \frac{\beta_k^*\beta_k}{2}\right) - \langle 1|\beta_k^* \tag{59}$$

operating on the left of (59) with $\hat{b}_k^\dagger$

$$\langle\beta_k|\hat{b}_k^\dagger = \langle 0|\hat{b}_k^\dagger - \langle 0|\frac{\beta_k^*\beta_k}{2}\hat{b}_k^\dagger - \langle 1|\beta_k^*\hat{b}_k^\dagger = \langle 0|\frac{\beta_k^*\hat{b}_k^\dagger\beta_k}{2} + \langle 1|\hat{b}_k^\dagger\beta_k^* = \langle 0|\beta_k^* \tag{60}$$

In (59) exchange the order of $\beta_k^*\beta_k$, multiply from the left by β_k^* and apply the definition in (50) to get the conjugate of (56)

$$\langle\beta_k|\beta_k^* = \langle 0|\beta_k^* \tag{61}$$

Equation (61) along with (60) will give the conjugate of (57)

$$\langle\beta_k|\hat{b}_k^\dagger = \langle\beta_k|\beta_k^* \tag{62}$$

To construct a multimode state one needs to apply successive operators to the vacuum state

$$|\rangle_F \equiv |\beta_1\beta_2\cdots\rangle = \prod_{k=1}^{\infty}\mathrm{e}^{\hat{b}_k^\dagger\beta_k - \beta_k^*\hat{b}_k}|0\rangle \tag{63}$$

which by means of the Hausdorff-Baker-Campbell relation and the commutator

$$[\hat{b}_i^\dagger\beta_i - \beta_i^*\hat{b}_i, \hat{b}_j^\dagger\beta_j - \beta_j^*\hat{b}_j] = 0$$

may be written regardless of the order in which the β_k's appear

$$|\rangle_F = \mathrm{e}^{\sum_{k=1}^{\infty}\hat{b}_k^\dagger\beta_k - \beta_k^*\hat{b}_k}|0\rangle \tag{64}$$

this means that, unlike the eigenstates with complex numbers [18], the multimode eigenstate with grassmann numbers may be used to represent physical states. Now that the multimode Fermion eigenstate is defined one may write the eigenvalue equation for the Fermionic field operator $\hat{\psi}(x)$. Clearly, the permutation property which is expressed by (64) will make it possible to write an eigenvalue equation for the kth mode using the multimode statevector

$$\hat{b}_k|\rangle_F = \beta_k|\rangle_F \tag{65}$$

with the expression

$$\hat{\psi}(x) = \sum_k \hat{b}_k\phi_k(x)$$

and (65) one may write the analogue of (8) for anticommuting fields

$$\hat{\psi}(x)|\rangle_F = \psi(x)|\rangle_F \tag{66}$$

however, now the eigenvalue

$$\psi(x) = \sum_k \beta_k \phi_k(x) \tag{67}$$

is a grassmann field. It is now possible to introduce the normalized fermionic amplitude function analogous to (10)

$$\Psi_F(x) = \frac{1}{\sqrt{N}}\, {}_F\langle|\hat{\psi}(x)|\rangle_F = \frac{1}{\sqrt{N}}\psi(x) \tag{68}$$

The average occupation number is

$$\langle n_k\rangle = \langle\beta_k|\hat{b}_k^\dagger \hat{b}_k|\beta_k\rangle = \beta_k^* \beta_k \tag{69}$$

where $\langle\beta_k|\beta_k\rangle = 1$ as one may easily verify by taking the product of (53) with (59). Furthermore with (65) and it's adjoint the number operator may also be calculated using the multimode eigenvector $\langle n_k\rangle = {}_F\langle|\hat{b}_k^\dagger \hat{b}_k|\rangle_F = \beta_k^*\beta_k$ leading to

$$\beta_k = \int d\beta_k^* \langle n_k\rangle \tag{70}$$

where for grassmann numbers the process of integration is equivalent to the process of taking the derivative, that is,

$$\int d\beta_k^* \beta_k^* = 1$$

With (65) in mind, one can relate the fermionic wavefunction in (68) to the average occupation number per mode

$$\begin{aligned}\Psi_F(x) &= \frac{1}{\sqrt{N}}\, {}_F\langle|\hat{\psi}(x)|\rangle_F = \frac{1}{\sqrt{N}}\, {}_F\langle|\sum_k \hat{b}_k\phi_k(x)|\rangle_F \\ &= \frac{1}{\sqrt{N}}\sum_k {}_F\langle|\hat{b}_k|\rangle_F \phi_k(x) = \frac{1}{\sqrt{N}}\sum_k \beta_k\phi_k(x)\end{aligned} \tag{71}$$

Equation (70) will directly give that relation

$$\Psi_F(x) = \frac{1}{\sqrt{N}}\sum_k \left(\int d\beta^* \langle n_k\rangle\right)\phi_k(x) \tag{72}$$

Equation (72) for anticommuting fields is analogous to (17) which was developed for commuting fields. The complex conjugate of (72) is given by

$$\Psi_F^*(x) = \frac{1}{\sqrt{N}}\, {}_F\langle|\hat{\psi}^\dagger(x)|\rangle_F = \frac{1}{\sqrt{N}}\sum_k \beta_k^*\phi_k^*(x) \tag{73}$$

and since $\beta_k^* = \int d\beta_k \langle n_k \rangle$ we may write (73) as

$$\Psi_F^*(x) = \frac{1}{\sqrt{N}} \sum_k \left(\int d\beta \langle n_k \rangle \right) \phi_k^*(x) \tag{74}$$

Integrating the product of (72) with (74) will prove that $\Psi_F(x)$ satisfies the normalization axiom of the probability density

$$\int d^3x \; \Psi_F^*(x) \Psi_F(x) = \frac{1}{N} \sum_j \int d\beta_j \langle n_j \rangle \int d\beta_j^* \langle n_j \rangle \tag{75}$$

where the orthonormal property of the set $\{\phi_k(x)\}$ have been utilized. Note that from (70)

$$\begin{aligned} \int d\beta_j \langle n_j \rangle &= \beta_j^* \\ \int d\beta_j^* \langle n_j \rangle &= \beta_j \end{aligned} \tag{76}$$

therefore

$$\int d\beta_j \langle n_j \rangle \int d\beta_j^* \langle n_j \rangle = \beta_j^* \beta_j = \langle n_j \rangle \tag{77}$$

Equation (77) together with $\sum_j \langle n_j \rangle = N$ proves that the right hand side of (75) is unity. Before proceeding further it is best to see a proof of the constraint $\sum_j \langle n_j \rangle = N$ when one is dealing with grassmann numbers. As in (21) of Sect. 2, consider the fermionic operator

$$\hat{n}(x) = \hat{\psi}^\dagger(x) \hat{\psi}(x) \tag{78}$$

with (66) and a density operator $\rho = |\rangle_F \, {}_F\langle|$ an ensemble average of $\hat{n}(x)$, similar to (1), will give

$$\langle \hat{n}(x) \rangle = \psi^*(x) \psi(x) = \sum_{k,k'} \beta_{k'}^* \beta_k \phi_{k'}^*(x) \phi_k(x) \tag{79}$$

where $\psi(x)$ is defined in (67). The total number of particles (quanta), or the total number of runs, is

$$N = \int d^3x \langle \hat{n}(x) \rangle = \sum_k \beta_k^* \beta_k = \sum_k \langle n_k \rangle \tag{80}$$

where the orthonormal property of the expansion basis in (67) have been employed, and where (77) have been used to write the final answer in the form of a sum over averages. Equation (80) shows that the normalization constraint applies to the grassmann field as well.

With the eigenstate of the anticommuting annihilation operator defined it is now possible to discuss the ensemble quantum probability of relativistic electrons. Relativistic electrons are particles which obey the spin-$\frac{1}{2}$ Dirac equation [19]

$$i\hbar \frac{\partial \bar{\Psi}(x)}{\partial t} = \left(c\hat{\bar{\alpha}} \cdot \hat{\bar{p}} + m_0 c^2 \hat{\beta} \right) \bar{\Psi}(x) \tag{81}$$

where $\bar{\Psi}(x)$ is a four component spinor. Since the Dirac field $\hat{\psi}(x)$ satisfies the Dirac field equation [8]

$$i\hbar\frac{\partial\hat{\psi}_k(x)}{\partial t} = \left(c\hat{\vec{\alpha}}\cdot\hat{\vec{p}} + m_0c^2\hat{\beta}\right)_{k\ell}\hat{\psi}_\ell(x) \tag{82}$$

the eigenvalue $\psi_k(x)$ of the eigenvalue equation

$$\hat{\psi}_k(x)|\rangle_F = \psi_k(x)|\rangle_F \tag{83}$$

satisfies the Dirac equation as well which means that the eigenvalue which appears in (83) is the kth solution to the Dirac equation for electrons. The relativistic electron wavefunction which satisfies the Dirac equation may therefore be written as

$$\Psi_k^{(+)}(x) = \frac{1}{\sqrt{N}}\psi_k^{(+)}(x) = \frac{1}{\sqrt{N}}\ {}_F\langle|\hat{\psi}_k^{(+)}(x)|\rangle_F \tag{84}$$

where the plus sign stands for positive energy solution to (81). The field operator for this particular solution of positive energy is given by

$$\hat{\psi}^{(+)}(x) = \int\frac{d^3p}{(2\pi\hbar)^{3/2}}\sqrt{\frac{m_0c^2}{E}}\sum_s\hat{b}(p,s)\,u(p,s)\mathrm{e}^{-\frac{i}{\hbar}p_\mu x^\mu} \tag{85}$$

where $p_\mu = (p_0, p_1, p_2, p_3) = (E/c, -p_x, -p_y, -p_z)$ is the covariant momentum four vector, $x^\mu = (x^0, x^1, y^2, z^3) = (ct, x, y, z)$ is the contravariant position four vector, $p_\mu x^\mu = Et - \vec{p}\cdot\vec{x}$ is the four vector scalar product, and where the index s in the sum stands for summing over spin orientation. The $u(p,s)$ is a bispinor, and $\hat{b}(p,s)$ is the electron annihilation operator. When operating on the Fermionic coherent state the operator $\hat{\psi}(x)$ will produce it's eigenvalue as it is shown in (65) and (66). The analogue of (17) for a relativistic electron can now immediately be written

$$\Psi^{(+)}(x) = \frac{1}{\sqrt{N}}\ {}_F\langle|\hat{\psi}^{(+)}(x)|\rangle_F = \int\frac{d^3p}{(2\pi\hbar)^{3/2}}\sqrt{\frac{m_0c^2}{NE}}\sum_s\beta(p,s)\,u(p,s)\,\mathrm{e}^{-\frac{i}{\hbar}p_\mu x^\mu} \tag{86}$$

where $\beta(p,s) = \int d\beta^*(p,s)\langle n(p,s)\rangle$, and where

$$N = \int d^3p\,\sum_s\langle n(p,s)\rangle \tag{87}$$

is the total number of particles with positive energy and both orientations of spin, and where $\beta(p,s)$, as in (70), is expressed in terms of average occupation number.

To test the normalization axiom of the probability function in (86) multiply that equation by it's adjoint and integrate over all space

$$\begin{aligned}\int d^3x\Psi^{(+)\dagger}(x)\Psi^{(+)}(x) &= \int\frac{d^3x}{N}\frac{d^3p'd^3p}{(2\pi\hbar)^3}\frac{m_0c^2}{\sqrt{E'E}}\sum_{s,s'}\beta^*(p',s')\beta(p,s)\\ &\quad\times u^\dagger(p',s')u(p,s)\mathrm{e}^{\frac{i}{\hbar}(p'_\mu-p_\mu)x^\mu}\\ &= \int\frac{d^3p'd^3p}{N}\delta_{\vec{p}',\vec{p}}\frac{m_0c^2}{\sqrt{E'E}}\sum_{s,s'}\beta^*(p',s')\beta(p,s)\\ &\quad\times u^\dagger(p',s')u(p,s)\mathrm{e}^{\frac{i}{\hbar}(p'_0-p_0)x^0}\end{aligned} \tag{88}$$

where [19]

$$\int d^3x\, \frac{e^{\frac{i}{\hbar}(\vec{p}-\vec{p}')\cdot\vec{x}}}{(2\pi\hbar)^3} = \delta_{\vec{p}',\vec{p}}$$

The integration over $\vec{p}'$ will make $p'_0 = p^0$, and that equality is due to the relativistic energy being $E = \sqrt{|\vec{p}|^2c^2 + (m_0c^2)^2}$, therefore

$$\int d^3x \Psi^{(+)\dagger}(x)\Psi^{(+)}(x) = \int \frac{d^3p}{N}\frac{m_0c^2}{E}\sum_{s,s'}\beta^*(p,s')\beta(p,s)u^\dagger(p,s')u(p,s) \tag{89}$$

which can be further simplified due to the relation [19]

$$u^\dagger(p,s')u(p,s) = \delta_{s',s}\left(\frac{E}{m_0c^2}\right)$$

leading to

$$\int d^3x \Psi^{(+)\dagger}(x)\Psi^{(+)}(x) = \frac{1}{N}\int d^3p \sum_s \beta^*(p,s)\beta(p,s) = \frac{1}{N}\int d^3p \sum_s \langle n(p,s)\rangle = 1 \tag{90}$$

where in the last step (77) and (87) were employed. It is possible to carry out the same calculations that lead to (90) for the negative energy solution of the Dirac equation. Equation (90) shows the consistency of $\Psi^{(+)}(x)$ with the axiom of probability and therefore demonstrates the possibility of extending the ensemble quantum probability to include relativistic electrons. However, the elements $\Psi^{(+)}(x)$ are grassmann field which was necessary because $\Psi^{(+)}(x)$ was defined to be the eigenvalue of the anticommuting annihilation field operator. The ensemble interpretation of the relativistic spin-$\frac{1}{2}$ electron is evident in the expression given by (86).

6 Concluding Remarks

The expansions of the wavefunctions in (17) and (86) suggest that the electron state have a dependence on an average of a quantity, and by having that dependence the electron state-function is including information about a possible fluctuation in measurements when conducting an experiment on a quantum particle in the void vacuum. In the re derivation of (17) for the non relativistic spinless electron which satisfies the exclusion principle a sequence of measurements, (42), were considered such that in each measurement an electron was assumed to be either occupying a kth state or not. The sequence in (42) described a situation in which for a total of j_k measurements n_k electrons were separately[10] found to occupy the kth state, while the $(j_k - n_k)$ measurements did not detect any electron in the kth state. Moreover (43) takes into account the total number of ways of distributing the n_k detections of an electron among the j_k measurements which means that the formalism that lead to (17)

[10] The word separately here is meant to indicate stochastic independence such that finding a kth state electron in the ith run of the experiment does not influence a similar event in the following experiments, and is not influenced by preceding experiments either. This is an important point which was brought up [5] in the SPM approach to stochastic quantum mechanics.

for fermions had implied in it a random electron detection process. In this sense the general state vector in (46) implies a stochastic process, one which is needed to formulate what David Bohm [20] calls the unpredictable and uncontrollable fluctuations in a measurement made on a quantum particle.

The stochastic approach to quantum phenomena was previously suggested by Nelson [21], Kershaw [22], and Rosenstein [5]. Stochastic mechanics [21] was used to treat the Schrödinger electron as a diffusing particle with a diffusion coefficient constant of $\hbar/2m$ where $\hbar$ is the Plank constant and where m is the electron mass. This approach lead to a sound derivation of the Schrödinger equation but also raised the question about the physical cause which makes the electron "diffusion" be comparable to that of a Brownian particle. Nevertheless the derivation of the Schrödinger equation from a stochastic process perspective can be taken as a very strong evidence that random fluctuations in the particle's state is a physical effect that may not be ignored. One is thus naturally lead to the question of what possible cause could exist in the void vacuum which can be responsible for the fluctuations in the measurement which in turns will give an ensemble state function as in (17). Nelson discusses that Bohm and vigier hypothesized a subquantum medium which by means of interaction with the electron causes the random fluctuations. Their idea of interactions causing fluctuating measurements is physically plausible but the hypothesis about the existence of a subquantum medium must have appeared as purely speculative by then. However, given the predictions of the quantum field theory (QFT) about a fluctuating vacuum and given the hypothesis of the zero point field (ZPF) the answer to the question about a cause to a measurement fluctuations does not become a puzzle anymore. For due to QFT and ZPF the vacuum is no longer the inactive void which do not permit any interactions with a quantum particle. The physical soundness of the existence of the ZPF is not speculative as many physical results confirmed by quantum theory were derived[11] from the hypothesis of a vacuum filled with a fluctuating electromagnetic fields and as experimental measurements of the detectable Casimir force [27] authenticates the hypothesis of the ZPF. Thus the sequence introduced in (42) is physically justified as one may not control the interaction of the electron with the fluctuating field. Therefore each time the experiment is conducted the electron may or may not be found in the kth state depending on how much energy it received from the fluctuating field it is submerged in. In fact the appearance of $\langle n_k \rangle$ in (17) indicates the impossibility of finding a quantum particle in a definite k state each time the experiment is conducted, a description which may be taken as a theoretical evidence for the existence of random forces on the electron from the vacuum. Supportive of this view is Boyer's statement that "in a sense, quantum motions are the experimental evidence for the existence of the zero point radiation" [28] to which we would like to add the statement that the successful prescription of the square of the wavefunction as a probability density is the theoretical evidence for the existence of the zero point radiation manifested by the appearance of $\langle n_j \rangle$ in (17). The physical existence of the ZPF plays an important role in as far as the cause of the experimental fluctuations that leads to the wave function's direct proportionality to the average particle number per state. Therefore a word about the origin of the hypothesized ZPF becomes relevant.

[11] These include: Puthoff's claim that stability of the ground state of hydrogen is a ZPF determined phenomenon. The calculations specifically show that the power absorbed by a circulating electron exactly equals to the power radiated due to it's acceleration. [23]; the treatment of the van der Waals forces as a ZPF determined force [24]; the recovery of the Heisenberg uncertainty principle for the harmonic oscillator where fluctuations in the electromagnetic field cause fluctuations in the positions of particles with electromagnetic interaction [25]; the recovery of the blackbody radiation spectrum from the hypothesis of electromagnetic zero point energy [26].

There appears to be some physical similarity that the hypothesized ZPF and the cosmic microwave background radiation (CMBR) have in common. For one thing, both the ZPF and CMBR agree that space is not a void vacuum but is filled with electromagnetic radiation, although they might disagree about the range of the electromagnetic radiation's spectrum. Another interesting similarity between the hypothesized ZPF and CMBR is the fact that both assume the electromagnetic radiation to be isotropic, however, one should be careful about this particular similarity as the CMBR is supposed to be only approximately isotropic. Yet the most striking similarity between a CMBR result and the hypothesized ZPF is the fact that they both exhibit the blackbody radiation spectrum. When assuming the existence of a ZPF, Timothy Boyer (see footnote 11) was able to derive the blackbody radiation spectrum by a theoretical argument similar to the one introduced first by Einstein and Hopf [29]. Boyer, however, imposed a new boundary condition which helped showing that the ZPF is responsible for the blackbody's spectrum departure from the Rayleigh-Jeans law. These semi agreements between the ZPF predictions and the CMBR results may be considered as a compelling evidence that the CMBR is in fact the hypothesized ZPF that many physicists are assuming to exist a priori. Since CMBR's results are undisputable any disagreement that results from a comparison between CMBR and ZPF should cause a correction or a remodeling of the ZPF such that it is completely consistent with the results of CMBR.

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